Assignment 2

Deadline

Friday 17th February, 2017

1. Suppose that p_1, p_2, p_3 are distinct primes and that $n, k \in \mathbb{Z}^+$ with $n = p_1^5\, p_2^3\, p_3^k$. Let A be the set of positive integer divisors of n and define a relation $\mathcal{R}$ on A by $x\mathcal{R}y$ if x exactly divides y. If there are 5880 ordered pairs in $\mathcal{R}$, determine k and $|A|$.

2. Let A be a set with $|A| = n$, and let $\mathcal{R}$ be an equivalence relation on A with $|\mathcal{R}| = r$. Why is $r - n$ always even?

3. A relation $\mathcal{R}$ on a set A is called *irreflexive* if for all $a \in A, (a, a) \notin \mathcal{R}$. Let $\mathcal{R}$ be a non-empty relation on A. Prove that if $\mathcal{R}$ satisfies two of the following properties - reflexive, symmetric, transitive, then it cannot satisfy the third.

4. Given a set A with n elements and a relation $\mathcal{R}$ on A, let M denote the relation matrix for $\mathcal{R}$. Then, prove the following:
   (a) $\mathcal{R}$ is *reflexive* iff $I_n \leq M$.
   (b) $\mathcal{R}$ is *symmetric* iff $M = M^T$
   (c) $\mathcal{R}$ is transitive iff $M.M = M^2 \leq M$

5. Prove that $M(\mathcal{R}) = \mathbf{0}$ iff $\mathcal{R} = \phi$.

6. Prove that $M(\mathcal{R}) = \mathbf{1}$ iff $\mathcal{R} = A \times A$.

7. Prove that $M(\mathcal{R})^n = [M(\mathcal{R})]^n$, for all $n \in \mathbb{Z}^+$.

8. Let $f : A \to B$. If $B_1, B_2 \ldots B_n$ is a partition of B, prove that $\{f^{-1}(B_i) | 1 \leq i \leq n, f^{-1}(B_i) \neq \phi\}$ is a partition of A.

9. Suppose that $\mathcal{R}$ and $\mathcal{S}$ are reflexive relations on a set A. Prove or disprove each of these statements:
   (a) $\mathcal{R} \cup S$ is reflexive
   (b) $\mathcal{R} \cap S$ is reflexive
   (c) $\mathcal{R} - S$ is irreflexive $\mathcal{R} \circ S$ is reflexive

10. Suppose that the relation $\mathcal{R}$ is irreflexive, is $\mathcal{R}^2$ necessarily irreflexive? Give reasons.

11. Let $\mathcal{R}$ be the relation on the set of all metro stations in Delhi, such that $(a, b) \in \mathcal{R}$ if it is possible to go from stop a to stop b without changing trains. What is $\mathcal{R}^n$, for a positive integer n?

12. Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that sR_nt if and only if $s = t$, or both s and t have at least n characters and the first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t begins with the n characters at the start of s. For example, let $n = 3$ and let S be the set of all bit strings. Then sR_3t either when $s = t$ or both s and t are bit strings of length 3 or more that begin with the same three bits. For instance, $01R_301$ and $00111R_300101$, but $01 \not R_3 010$ and $01011 \not R_3 01110$. Show that for every set S of strings and every positive integer n, R_n is an equivalence relation on S.

13. Let R_3 be the relation from previous question. What are the sets in the partition of the set of all bit strings arising from the relation R_3 on the set of all bit strings?

14. Each bead on a bracelet with three beads is either red, white, or blue. Define the relation $\mathcal{R}$ between bracelets as: $(B_1, B2)$, where B_1 and B_2 are bracelets, belongs to $\mathcal{R}$ if and only if B_2 can be obtained from B_1 by rotating it or rotating it and then reflecting it.

    (a) Show that $\mathcal{R}$ is an equivalence relation.

    (b) What are the equivalence classes of $\mathcal{R}$?

15. How many equivalence relations are there over the set A = (a, b, c)?

16. Given the partition P = 1, 2, 3, 4, 5 of the set A = 1, 2, 3, 4, 5, consider R the associated equivalence relation on A. Draw the digraph associated to R and write down the matrix M(R).

17. Prove that if R is a relation and S $\subseteq$ R, then S is a relation.

18. If R is a reflexive relation on S, then so is any superset of R inside S $\times$ S.

19. The following problems pertain to the relationship of congruence mod n, defined on Z as follows: DEFINITION: Let a and b be integers and let n be a positive integer. Then a $\equiv$n b iff n | (a $-$ b). Show that 2 | (x$-$y) iff x and y have the same parity; i.e., either both x and y are even or both are odd.

20. Determine whether the following relations are reflexive, symmetric, or transitive. Prove your claims. D = (x, x) : x $\in$ S, the diagonal of S $\times$ S, where S is any set.