# Assignment 1

Friday $27^{\text{th}}$ January, 2017

1. Given a function $f : X \to X$ such that $f(x+y) = f(xy)$ when:
   (a) $x, y \geq 1$ and $X = 1, 2, 3, \ldots$
   (b) $x, y \geq 4$ and $X = 8, 9, 10, \ldots$

   Find the value of $f(9)$ when $f(8) = 9$. (* *)

2. Count the number of rectangles present in a rectangular grid consisting of 100 rows and 95 columns. (*)

3. Give a story proof as well as a mathematical proof for the following:

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k} \qquad (**) \qquad (1)$$

4. Find the value of the following. Provide a story proof used in determining the values: (* *)

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{n} \qquad (2)$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots \binom{n}{n}^2 \qquad (3)$$

5. Six boxes are colored red, black, blue, yellow, orange and green. In how man ways can you put 20 identical balls into these boxes such that no box is empty? (* *)

6. A playground has 4 see-saws placed right next to each other. 31 students from a class visit the playground and wish to play on the see-saw. How many ways can the students be placed on the see-saw if 10 of he students wish to sit only on the left side, 12 on the right side and remaining 9 can sit on either side? (* * *)

7. In how many ways can you triangulate a regular polygon having $n + 2$ sides? (* *)

8. Check if $(n^2 + n + 41)$ always results in a prime, for all positive integers $n$. If yes, prove using induction. (*)

9. Prove that $\binom{n}{k} p^k (1-p)^{n-k} \leq 1$ for any $n, k \in \mathbb{N}$ and $0 \leq p \leq 1$.

10. Given a $2^n \times 2^n$ chess board with one square removed, show that you can tile the chessboard using triominoes (L shaped dominoes). (* * *)

11. Prove that $\left(\sum_{k=0}^{n} \binom{n}{n-k} x^{2k-n} \geq 2^n\right)$, for $n \in \mathbb{N}$ and $x \in \mathbb{R}^+$. (*)

12. Let $x + 1/x$ be an integer for $x \in \mathbb{R}^+$. Prove that $x^n + 1/x^n$ is an integer for $n \in \mathbb{N}$. (* *)

13. There are 8 guests at a party and they sit around an octagonal table with one guest at each edge. If each place at the table is marked with a different person's name and initially everybody is sitting in the wrong place, prove that the table can be rotated in such a way that at least 2 people are sitting in the correct places. (* *)

14. There are n people present in a room. Prove that among them there are two people who have the same number of acquaintances in the room. (* *)

15. In any group of six people, prove that there are either 3 mutual friends or 3 mutual strangers. Is the same true when the group has 5 people? (* * *)

16. A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white balls. What is the minimum no. of balls we have to choose randomly from the box to ensure that we get 9 balls of same color?

17. How many ways are there to assign five different jobs to four different employees if every employee is assigned at least one job? (*)

18. We are given the plot of a function $f$. (*)

    (a) There exists a horizontal line that intersects the plot at three places. What can we say about the type of this function? Give an example of such a function.

    (b) There exists a vertical line that does not intersect the plot of the graph. What can we say about the type of this function? Give an example of such a function.

19. A binary relation can have the properties of reflexive, symmetric and transitive. Thus we have 8 possible different subsets of these properties a binary relation can have. Give an example of a relation in each of the eight cases. (* *)

20. How many relations are there on a set with $n$ elements that are: (* *)

    (a) Symmetric

    (b) Antisymmetric

    (c) Asymmetric

    (d) Irreflexive

    (e) Reflexive and symmetric

    (f) Neither reflexive nor irreflexive