Assignment 7
Solutions of Generating Functions

1. $c_1 + c_2 + c_3 + c_4 = 20$ where $-3 \leq c_1, c_2, -5 \leq c_3 \leq 5, 0 \leq c_4$
   $(3 + c_1) + (3 + c_2) + (5 + c_3) + c_4 = 31$
   By replacing the variables now the problem turns into $x_1 + x_2 + x_3 + x_4 = 31$ where $0 \leq x_1, x_2, x_4; 0 \leq x_3 \leq 10$
   Hence the answer is the coefficient of x^{31} in the generating function :
   $(1 + x + x^2 \ldots)^3(1 + x + x^2 .... + x^{10})$.

2. Using the idea of generating functions we have :
   (a) we would have $(x^3 + x^4 + ...)^4$ but we can take x^{12}as a common factor so we have $(x^3 + x^4 + ...)^4 = x^{12}(1 + x + x^2 + ...)^4$.
   We also know that $\frac{1}{1-x} = (1 + x + x^2 + ...)$ and so $\frac{1}{(1-x)^4} = (1 + x + x^2 + ...)^4$ and so we have $x^{12}(1 + x + x^2 + ...)^4 = x^{12}(1 - x)^{-4}$. Now want to find the coefficient of x^{12} in $(1 - x)^{-4}$ which is $\binom{-4}{12}(-1)^{12} = (-1)^{12}\binom{4+12-1}{12} = \binom{15}{12}$
   (b) In a similar way, we need to find the coefficient of x^{12} in $(1 + x + x^2 + ... + x^6)^4$.

3. Consider each package of 25 envelopes as one unit. Then the answer is the coefficient of x^{120} in $(x^6 + x^7 + \ldots + x^{39} + x^{40})^4 = x^{24}(1 + x + \ldots + x^{34})^4$, which is the same as the coefficient of x^{96} in $(\frac{1-x^{35}}{1-x})^4$

4. There is a one-one correspondence between the possible subsets and the solutions of the equation $c_1 + c_2 + c_3 + \ldots c_8 = 49$, where $c_1, c_8 \geq 0, c_i \geq 0 \; \forall \; 2 \leq i \leq 7$.
   The number of these solutions is the coefficient of x^{49} in the generating function :
   $(1 + x + x^2 \ldots)(x^2 + x^3 + \ldots)^6(1 + x + x^2 \ldots) = x^{12}/(1 + x)^8$.
   This can be seen as the coefficient of x^37 in $(1 - x)^{-8}$ which is equal to $\binom{44}{37}$.

5. The number of partitions of 6 into 1's 2's and 3's is 7.

6. Let $a(x)$ be the generating function for number of partitions of n where no summand appears more than twice and $b(x)$ be the generating function for number of partitions of n where no summand is divisible by 3. It suffices to show that $a(x)$ and $b(x)$ are the same.
   Observe that the generating function for $a(n)$ is given by
   $a(x) = (1 + x + x^2)(1 + x^2 + x^4)(1 + x^3 + x^6) \ldots = \frac{1-x^3}{1-x} \cdot \frac{1-x^6}{1-x^2} \cdot \frac{1-x^9}{1-x^3} \ldots = b(x)$ where,
   $b(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$

7. Let $f(x)$ be the generating function for number of partitions of n where no summand is divisible by 4 and $g(x)$ be the generating function for number of partitions of n where no even summand is repeated. It suffices to show that $f(x)$ and $g(x)$ are the same.
   $f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$
   $g(x) = \frac{1}{1-x}.(1 + x^2).\frac{1}{1-x^3}.(1 + x^4).\frac{1}{1-x^5}.(1 + x^6) \ldots$
   $= \frac{1}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1-x^6}{1-x^4} \cdot \frac{1}{1-x^5} \cdot \frac{1-x^{12}}{1-x^6} \cdots$
   $= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots = f(x)$

8. We can consider the Ferrers graph with summands (rows) not exceeding m. Now when we consider the transpose, we obtain yet another Ferrers graph that has m summands (rows). The result follows from the one-one correspondence of the between these graphs.

9. the exponential of the sequence 0!,1!,2!.............. is given by $\frac{0!}{0!}x^0 + \frac{1!}{1!}x^1 + \frac{2!}{2!}x^2$.........
   $= 1 + x^1 + x^2 + x^3 .... = \frac{1}{1-x}$