

Solutions for Assignment 6

Principle of Inclusion and Exclusion

1. We must find positive integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5 = 15$, where $1 \leq x_i \leq 4$ for all $1 \leq i \leq 5$.

This is equal to the non-negative integer solutions for $y_1 + y_2 + y_3 + y_4 + y_5 = 10$, where $0 \leq y_i \leq 3$ for all $1 \leq i \leq 5$.

We denote the condition c_i for $1 \leq i \leq 5$ as the solutions where $y_i > 3$ and $y_j \geq 0$ for $1 \leq j \leq 5$ and $i \neq j$. Therefore, to find solutions for $N(c_1)$ we would find the positive integer solutions for the equation $z_1 + z_2 + z_3 + z_4 + z_5 = 6$, where $z_1 + 4 = y_1; z_i = y_i$ for all $2 \leq i \leq 5$.

Similarly, for $N(c_i c_j)$ we would find the positive integer solutions for the equation $w_1 + w_2 + w_3 + w_4 + w_5 = 2$, where $w_i + 4 = y_i; w_j + 4 = y_j; z_k = y_k$ for all $1 \leq k \leq 5; k \neq i, j$.

We determine the solution as :

$$\begin{aligned} N(\overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4} \overline{c_5}) &= S_0 - S_1 + S_2 - S_3 + S_4 - S_5 \\ &= \binom{5+10-1}{10} - \binom{5}{1} \binom{10}{6} + \binom{5}{2} \binom{6}{2} - 0 + 0 - 0 \\ &= \binom{14}{10} - \binom{5}{1} \binom{10}{6} + \binom{5}{2} \binom{6}{2} \end{aligned}$$

However, for each such arrangement, we have $15!$ ways of placing the muffins by permuting them. Hence the total arrangements possible would be $15!$ times the above solution.

2. We denote c_i to denote the condition that i did not appear in any of the dice. Now all we have to determine is $N(\overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4} \overline{c_5} \overline{c_6})$. Using the principle of inclusion and exclusion, we have the solution as below:

$$[6^8 - \binom{6}{1}5^8 + \binom{6}{2}4^8 - \binom{6}{3}3^8 + \binom{6}{4}2^8 - \binom{6}{5}1^8]/6^8$$

The division by 6^8 to divide by the total sample space.

3. For $1 \leq i \leq 7$, let c_i denote the situation where the i^{th} friend was at lunch with her. We need to determine if $N(\overline{c_1} \overline{c_2} \overline{c_3} \dots \overline{c_7})$ is positive or not.

$$\begin{aligned} N(\overline{c_1} \overline{c_2} \overline{c_3} \dots \overline{c_7}) &= 84 - \binom{7}{1}35 + \binom{7}{2}16 - \binom{7}{3}8 + \binom{7}{4}4 - \binom{7}{5}2 + \binom{7}{6}1 - \binom{7}{7}0 \\ &= 0 \end{aligned}$$

Hence we can conclude that she always had company for lunch.

4. Let c_1 denote the presence of consecutive E's in the arrangement, likewise c_2, c_3, c_4, c_5 are for N's O's R's S's respectively.

- (a) no consecutive identical letters

$$\begin{aligned} N(\overline{c_1} \overline{c_2} \overline{c_3} \dots \overline{c_5}) &= S_0 - S_1 + S_2 - S_3 + S_4 - S_5 \\ &= 14!/(2!)^5 - \binom{5}{1}[13!/(2!)^4] + \binom{5}{2}[12!/(2!)^3] - \binom{5}{3}[11!/(2!)^2] + \binom{5}{4}[10!/(2!)] - 9! \end{aligned}$$

- (b) If E_m denotes the number of elements in the set that satisfy precisely m conditions out of t condition, we have :

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} \dots (-1)^{t-m} \binom{t}{t-m} S_t$$

$$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 - \binom{5}{3} S_5$$

- (c) If L_m denotes the number of elements in the set that satisfy at least m conditions out of t condition, we have :

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} \dots (-1)^{t-m} \binom{t-1}{m-1} S_t$$

$$L_3 = S_3 - \binom{3}{2} S_4 + \binom{4}{2} S_5$$

5. For $1 \leq i \leq 7$, c_i denote the condition that i is not in the range of f .
Functions where $|f(A)| = 4$ is given by:

$$\begin{aligned} E_3 &= S_3 - \binom{4}{1} S_4 + \binom{5}{2} S_5 - \binom{6}{3} S_6 + \binom{7}{4} S_7 \\ &= \binom{7}{3} 4^{10} - \binom{4}{1} \binom{7}{4} 3^{10} + \binom{5}{2} \binom{7}{5} 2^{10} - \binom{6}{3} \binom{7}{6} 1^{10} + \binom{7}{4} \binom{7}{7} 0^{10} \end{aligned}$$

For functions where $|f(A)| \leq 4$ we have $L_3 = S_3 - \binom{3}{2} S_4 + \binom{4}{2} S_5 - \binom{5}{2} S_6 + \binom{6}{2} S_7$

6. Given we model the problem as $1 \leq i \leq 4$ where c_i denotes void in clubs($i = 1$), diamonds($i = 2$), hearts($i = 3$), spades($i = 4$).

- (a) The probability that at least one from each suit is included is $= N(\overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4}) / \binom{52}{13} = \binom{52}{13} - \binom{4}{1} \binom{39}{13} + \binom{4}{2} \binom{26}{13} - \binom{4}{3} \binom{13}{13}$

- (b) The probability of exactly one void is $E_1 / \binom{52}{13}$ where $E_1 = S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3 - \binom{4}{3} S_4 = \binom{4}{1} \binom{39}{13} - 2 \binom{4}{2} \binom{26}{13} + 3 \binom{4}{3} \binom{13}{13} - 0$

- (c) The probability of exactly two void is $E_2 / \binom{52}{13}$ where $E_2 = S_2 - \binom{3}{1} S_3 = \binom{4}{2} \binom{26}{13} - 3 \binom{4}{3} \binom{13}{13}$

7. (a) (d_{10}^2) , where d_{10} denotes the number of derangements of 10 items.

- (b) From $1 \leq i \leq 10$ let c_i denote that goon i gets back both his possessions.

$$N(\overline{c_1} \overline{c_2} \overline{c_3} \dots \overline{c_{10}}) = (10!)^2 - \binom{10}{1} 9!^2 + \binom{10}{1=2} 8!^2 \dots (-1)^{10} \binom{10}{0} 0!^2$$

8. (a) $(12!)d_{12}$

(b) $12! \binom{12}{6} d_6$

9. We first construct the matrix that represents the students in rows and the subjects in columns and the forbidden assignments. We then write the rooks polynomial for the board where the rooks are considered to be placed on the forbidden slots as below:

$$r(C, x) = (1 + 4x + 3x^2)(1 + 4x + 2x^2) = 1 + 8x + 21x^2 + 20x^3 + 6x^4$$

For $1 \leq i \leq 5$, let c_i be the condition that an assignment is made with person i assigned to a language he/she wishes to avoid.

$$N(\overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4} \overline{c_5}) = 5! - 8(4!) + 21(3!) - 20(2!) + 6(1!) = 20$$

The above can be easily obtained by using the coefficients of the rook polynomial as $S_i = r_i(5 - i)!$ where r_i is the coefficient of x^i .

10. In this question, we see the benefit of rearranging the rows and columns of the matrix representing the forbidden arrangements.

Assume the rows to be in the order 1,5,3,4,2,6 and columns in the order 1,5,2,4,6,3.

Now when we mark the forbidden slots, we can see that the resulting board can be broken into smaller independent boards. Same as in the previous example, we construct the rook polynomial for arranging the rooks in the forbidden slots rather than constructing it for the fr

$$r(C, x) = (1 + 4x + 2x^2)(1 + 3x + 3x^2)(1 + x) = 1 + 8x + 22x^2 + 25x^3 + 12x^4 + 2x^5$$

For $1 \leq i \leq 6$, let c_i denote the condition that all 6 values appear on the red and green die

after rolling 6 times but i appeared on the red die paired with one of the forbidden numbers on the green.

$$N(\overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4} \overline{c_5} \overline{c_6}) = 6! - 8(5!) + 22(4!) - 25(3!) + 12(2!) - 2(1!) + 0(0!) = 160$$

Then the probability that every value came up on the die is: $(6!)(160)/[(28)^6]$

11. Here we have $r(C, x) = 1 + 8x + 20x^2 + 17x^3 + 4x^4$

Let the conditions be specified as follows:

$$c_1 : f(1) = v \text{ or } w$$

$$c_2 : f(2) = u \text{ or } w$$

$$c_3 : f(3) = x$$

$$c_4 : f(4) = v, x \text{ or } y$$

The solution to be posed problem is $N(\overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4}) = 6! - 8(5!) + 20(4!) - 17(3!) + 4(2!) = 146$