Solution (Major Exam)

May 16, 2017

Section I

1. There are two possible solutions- one assuming 0 can be placed in the first position and the other with 0 not possible in the first position.

   Solution 1 : First place can be filled up in 8 ways (excluding 1 and 0)
   Second place can be filled up in 9 ways (excluding 2)
   Third place can be filled up in 9 ways (excluding 3)
   Fourth place can be filled up in 9 ways (excluding 4)
   Total possible numbers $= 8 * 9 * 9 * 9 = 5832$

   Solution 2 : First place can be filled up in 9 ways (excluding 1)
   Second place can be filled up in 9 ways (excluding 2)
   Third place can be filled up in 9 ways (excluding 3)
   Fourth place can be filled up in 9 ways (excluding 4)
   Total possible numbers $= 9 * 9 * 9 * 9 = 6561$

2. We alternatively solve for the following equation -

   $$y_1 + y_2 + y_3 + y_4 = 6$$

   where $y_i \geq 0$. The positive integer solutions for this equation can be found as $\binom{6+4-1}{6} = 84$

3. Show that $k = 10$ satisfies the relation $11 | 2^k - 1$. Or, consider the set $S = \{2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11}\}$. By PHP we know that $11 | 2^i - 2^j$ for some $i, j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Since 11 and 2 are relatively prime we can say that $11 | 2^{i-j} - 1$.

4. Any valid example will do. For example consider $A = 1, 2$ and a relation on $R = \{(1, 1), (1, 2)\}$.

5. This holds if $A = B$.

6. Consider the *star*-graph which is an example of a tree and look at its complement. This provides a counter example.

7. The only possible non-isomorphic induced subgraphs possible on K_6 are - $K_0, K_1, K_2, K_3, K_4, K_5, K_6$. Hence we can say that it is possible to have 7 graphs including the order-zero graph.

8. You can either provide an example and validate with a truth table. alternatively, you can provide an logical expression equivalent to $p \implies q$ and validate the equivalence through the truth table. An example for the first possibility is shown below:
If you get an A, then I'll give you a prize.
P:you get an A
Q:I'll give you a prize.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

9. The rook polynomial of a 3×3 chess board is $6x^3 + 18x^2 + 9x + 1$.

10. The maximum length of a trail in K_{2n} is $\binom{n}{2} - \frac{1}{2}(n-2)$

Section II

1. We prove this by a contradiction. Let us assume that such an arrangement was in fact possible. Then, when we consider every alternate seat in the circular table (i.e seat 1, 3, 5, 7 etc..), no two boys must appear consecutively. Hence, the maximum number of boys that can appear on these seats are $\lfloor \frac{25}{2} \rfloor = 12$. A similar argument can be made by considering seat 2, 4, 6, 8 . . . 24. Thus, the maximum number of boys that can be seated on the table would be 24, contrary to the 25 boys available to be seated. Hence, such an arrangement is not possible.

2. A counting argument can be made, where we observe the number of times the degree of vertex v is counted on the L.H.S as well as in the R.H.S. Through this we establish the equivalence of the two expressions.

3. This can be shown by considering the given graph as union of three disjoint graphs - K_n complete graph that lies at the intersection, $G_1 - K_n$ and $G_2 - K_n$. Then solution can be seen from Grimaldi - theorem 11.14.

4. Let a(n) be the number of partitions of n where no summand (part) appears more than twice and b(n) be the number of partitions of n where no

summand is divisible by 3. It suffices to show that a(n) and b(n) have the same generating function. Observe that the generating function for a(n) is $((1+q^1+q^{(1+1)})(1+q^2+q^{(2+2)})(1+q^3+q^{(3+3)})........)$

$$= \prod_{i=1}^{\infty}(1+q^n+q^{2n})$$

Next,

$$= \prod_{i=1}^{\infty}(1+q^n+q^{2n}) = \frac{\prod_{i=1}^{\infty}(1-q^{3n})}{\prod_{i=1}^{\infty}(1-q^n)} = \frac{1}{\prod_{i=1}^{\infty}(1-q^{3n-1})(1-q^{3n-2})}$$

which is the generating function for b(n) and so we are done.

5. We consider any two matrices belonging to the set A, and show how the "precedes" relation is reflexive, anti-symmetric and transitive.