Indian Institute of Technology Ropar

Discrete Mathematical Structures

Major Exam - CSL101

May 9, 2018

Name: ....

Entry Number: ....

Instructions:

- Please read the questions carefully and answer all of them as there are no choices.
- The paper consists of 3 sections, each worth 8 marks. Thus the answer script will be evaluated out of 24 marks.
- Please restrict to answering the questions in the provided space only.
- Kindly note that no clarifications will be made in the exam hall. In the case of any doubt, please state your assumptions and supply the answer accordingly. Please mail the instructor regarding the query after the exam. Genuine cases will be considered.

Marks Distribution

Section - I

Question No.	1	2	3	4	5	6	7	8	Total
Marks obtained									

Section - II

Question No.	1	2	3	4	Total
Marks obtained					

Section - III

Question No.	1	2	Total
Marks obtained			

Grand Total = $\frac{\quad}{24}$

Section I [8× 1 mark = 8 marks]

1. For which sets A, B it is true that $A \times B = B \times A$?

   Solution: This is true only when the sets are the same, i.e $A = B$.
   Scheme :
   Boolean Marking scheme.

2. Let A, B be sets with $|B| = 3$. If there are 4096 relations from A to B, what is $|A| =$?

   Solution: Each relation can be seen as a subset of the set $A \times B$. Thus, if the number of elements in A are n, we have $|A \times B| = 3n$. Thus total possible subsets of this set is 2^{3n}.
   Since it is given that there are 4096 total relations we have: $2^{3n} = 4096$.
   Solving the above equation we obtain $n = 4$.
   Scheme :
   Boolean Marking scheme.

3. Write the sequence generated by the generating function $\dfrac{1}{1-x^2}$.

   Solution:
   $f(x) = \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 \dots$
   Scheme :
   Boolean Marking scheme.

4. Write definition of bijective function with an example.

   Solution:
   A function $f : A \to B$ is said to be bijective iff f is both one-one and onto function.
   Any valid example of the same will do.
   Scheme :
   Boolean Marking scheme.

5. If $|A| = |B| = 5$, how many functions $f : A \to B$ are invertible?

   Solution: **5!**
   Scheme : Boolean Marking scheme.

6. Let m, n be positive integers with $1 < n \leq m$. Then prove that $S(m+1, n) = S(m, n-1) + nS(m, n)$.

   Solution:
   We know that $S(m, n)$ is the Sterlings number and denotes the number of ways in which m parties can be split into n teams. Thus a story proof of the above relation can be given as below:
   The total number of ways in which $m+1$ parties are split into n teams consists of all those ways in which a particular party P_1 is kept alone in a team and the rest m parties are split into $n-1$ teams given by $S(m, n-1)$. Further, we account for all those ways in which the m parties excluding P_1 are split into n teams and later P_1 is included in one of the existing n teams. This can be obtained in a total of $nS(m, n)$ ways.

   Scheme :
   Boolean Marking scheme.

7. Prove or Disprove: If G is simple graph and $|E| = |V| - 1$, then G is connected.

   Solution: The above statement is not true. We prove this by a counter example. Consider a graph on 4 vertices and 3 edges. The graph can be disconnected with 1 isolated vertex and a complete graph on the remaining 3 vertices.
   Scheme :
   Boolean Marking scheme.

8. Give an example of $f : \mathbb{Z} \to \mathbb{Z}$ where
   (i) f is one-to-one but not onto.
   (ii) f is onto but not one-to-one.

   Solution: Any valid example will do.
   Scheme :
   Boolean Marking scheme. 1 M is awarded only when both examples are given.

Section - II [$4\times$ 2 marks = 8 marks]

1. In how many ways can the 26 letters of the alphabet be permuted so that none of the patterns *car*, *dog*, *pun*, *byte* occurs?
   Solution:
   Let c_1 denote the condition where a permutation of the alphabet contains the word *car*, similarly c_2, c_3, c_4 be for *dog, pun, byte* respectively.
   Thus, we can say that $N(c_1) = N(c_2) = N(c_3) = 24!$ and $N(c_4) = 23!$
   Similarly when we consider two words at a time we obtain $N(c_1c_2) = N(c_1c_3) = N(c_2c_3) = 22!$ and we have for $1 \leq i \leq 3$ $N(c_ic_4) = 21!$.
   Now, considering 3 words at a time we have $N(c_1c_2c_3) = 20!$ and $1 \leq i < j \leq 3$ $N(c_ic_jc_4) = 19!$.
   Thus, overall the total number of permutations that do not contain any of the patterns is given by
   $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4}) \ = 26! - [3(24!) + 23!] + [3(22!) + 3(21!)] - [20! + 3(19!)] + 17!$
   Scheme:
   (a) Modeling the conditions correctly - 0.5 M
   (b) Part (a) is correct and the $N(c_ic_j...)$ are computed correctly - 0.5 M
   (c) Correct expansion of $N(\overline{c_i's})$ given - 0.5 M
   (d) Final answer computed correctly - 0.5 M

2. Let A be a set with $|A| = n$, and R be an equivalence relation on A with $|R| = r$. Why is $r - n$ always even?
   Solution:
   R is given to be an equivalence relation. Thus, for every $a \in A$, we must have $(a, a) \in R$. Since the cardinality of the set A is n, we are guaranteed to have n such pair of elements in R. Thus, $r - n$ counts the number of elements (a, b) in R such that $a \neq b$. Further, since R is also symmetric, for every pair $(a, b) \ni' a \neq b$, we have $(b, a) \in R$. This shows that every element in the remaining $r - n$ elements can be uniquely paired off to another element in the set. Thus $r - n$ must be even.
   Scheme:
   Boolean marking scheme.
   We can maybe consider giving 0.5 M to those who have at least mentioned what an equivalence relation is.

3. Determine the number of positive integers x, where $x \leq 9,999,999$ and the sum of the digits in x equals 31.
Solution:
Let the seven digits that make up x be $x_1, x_2, x_3, x_4, x_5, x_6, x_7$.
The number of solutions to the equation:
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 31$$
where $1 \leq i \leq 7$ and $0 \leq x_i \leq 9$ gives the answer to the posed question.
We use principle of inclusion and exclusion to compute the same.
Let c_i denote the condition that $x_i > 9$ in the above equation.
This can be obtained as the solution to the following equation: $y_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 21$ with $y_1, x_i \geq 0$.
Thus, $N(c_1) = \binom{27}{21}$ and therefore $\sum_{i=1}^{7} N(c_i) = \binom{7}{1}\binom{27}{21}$

Similarly, we obtain $N(c_1c_2)$ as the solution to $y_1 + y_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 11$ with $y_1, y_2, x_i \geq 0$.
Thus, $N(c_1c_2) = \binom{17}{11}$ and therefore $\sum_{i<j}^{7} N(c_ic_j) = \binom{7}{2}\binom{17}{11}$
$N(c_ic_jc_k) = \binom{7}{3}\binom{7}{1}$
It is easy to observe that it is not possible to satisfy more than 3 conditions taken at a time.
$N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4})\ \overline{c_6}\ \overline{c_7})\ ) = \binom{37}{31} - \binom{7}{1}\binom{27}{21} + \binom{7}{2}\binom{17}{11} - \binom{7}{3}\binom{7}{1}$
Scheme :
Boolean Marking scheme.

4. In how many ways can Troy select nine marbles from a bag of 12 (identical except for colour), where 3 are Red, 3 Blue, 3 White and 3 Green?
Solution:
We again solve this using P.I.E
In a similar manner as in the previous question, the number of possible ways is given by the solution to the equation :
$x_1 + x_2 + x_3 + x_4 = 9$, where $0 \leq x_i \leq 3$.
Here, x_1, x_2, x_3, x_4 denotes the number of red, blue, white and green marbles picked respectively.
In a similar manner as above we get
$N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4})\ ) = \binom{12}{9} - \binom{4}{1}\binom{8}{5} + \binom{4}{2}\binom{4}{1}$
Scheme :
Boolean Marking scheme.

Section - III **[2× 4 marks = 8 marks]**

1. Show that the number of partitions of a positive integer n, where no summand is divisible by 4 equals to the number of partitions of n where no even summand is repeated (although odd summands may or may not be repeated).
   Solution:
   Let $f(x)$ be the generating function for the number of partitions of n where no summand is divisible by 4. Then we know that :
   $f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \frac{1}{1-x^5} \cdot \frac{1}{1-x^6} \cdot \frac{1}{1-x^7} \frac{1}{1-x^9} \cdots$
   Similarly let $g(x)$ be the generating function for the number of partitions with no even summand repeated. Thus, we have
   $g(x) = \frac{1}{1-x} \cdot (1+x^2) \cdot \frac{1}{1-x^3} \cdot (1+x^4) \ldots$
   $= \frac{1}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdots$
   $= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \frac{1}{1-x^5} \cdot \frac{1}{1-x^6} \cdot \frac{1}{1-x^7} \frac{1}{1-x^9} \cdots$
   $= f(x)$
   Scheme :
   Giving the correct expansion of $f(x)$ - 1 M
   Giving the correct expansion of g(x) - 1 M
   Showing the reduction of $g(x)$ to $f(x)$ - 2 M

2. Given sequence $a_0, a_1, a_2, ...$ and $b_0, b_1, b_2, ...$ with exponential generating functions $f(x)$ and $g(x)$ respectively. Show that if $h(x) = f(x).g(x)$, then $h(x)$ is the exponential function of the sequence $c_0, c_1, c_2, ...$ where $c_m = \sum_{i=0}^{m} \binom{m}{i} a_i b_{m-i}$
   Solution:
   $h(x) = c_0 + c_1 x + c_2 \frac{x^2}{2!} + c_3 \frac{x^3}{3!} \ldots$
   But since we know that $h(x) = f(x)g(x)$, we can say that,
   $c_m(x^m/m!) = \sum_{i=0}^{m} (a_i x^i/i!)(b_{m-i}x^{m-i}/(m-i)!)$
   $= [\sum_{i=0}^{m} a_i b_{m-i}/(i!(m-i)!)]x^m$
   $= [\sum_{i=0}^{m} \{m!/i!(m-i)!\} a_i b_{m-i}](x^m/m!)$
   $= [\sum_{i=0}^{m} \binom{m}{i} a_i b_{m-i}](x^m/m!)$

   Thus we have $c_m = [\sum_{i=0}^{m} \binom{m}{i} a_i b_{m-i}]$

   Scheme :
   Giving the correct expansion of an exponential generating function - 1 M
   Correct derivation of c_m in terms of a_i's and b_j's - 3 M (Boolean Marking)

-x-x-x-x-x-x-x-