CSL105 : Discrete Mathematics

Minor Examination

Indian Institute of Technology Ropar

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Total Duration : 2 hours | Total Marks : 80 M

Section I [5 Marks each]

1. Let $S = \{1, 2, 3, \dots\}$. Consider a relation $R = \{(a, b) | a + b \leq 10\}$. Prove or disprove that R satisfies reflexive, symmetric, antisymmetric and transitive properties.

   Solution:

   (a) R cannot be reflexive since $\forall a \in S \ni' a \geq 6, (a, a) \notin R$

   (b) R is symmetric since $a + b = b + a$ and hence $\forall (a, b) \in R$, we can say that $(b, a) \in R$

   (c) R is not antisymmetric since $(8, 2) \in R$ and $(2, 8) \in R$ but $2 \neq 8$.

   (d) R is not transitive since, $(3, 2) \in R$ and $(2, 8) \in R$ but $(3, 8) \notin R$ since $3 + 8 \geq 10$.

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2. In the matrix representation of a relation, how does one find if the relation is transitive or not? Prove that your method works.
   Solution:
   Let R be transitive and M denote the matrix representing R.
   For R to be transitive, we know that $M^2 \leq M$

   Let M^2_{xy} be the the x^{th} row y^{th} column entry of M^2.
   If $M^2_{xy} = 1$ then there must exist at least one $y \in A \ni' M_{xy} = M_{yz} = 1$.
   This happens only if xRy and yRz guaranteeing that R is transitive.

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3. Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

   Solution:

   Let $a_1, a_2, \ldots a_{n^2+1}$ be the sequence of $n^2 + 1$ distinct real numbers.
   For $1 \leq k \leq n^2 + 1$, let x_k denote the maximum length of a decreasing subsequence that ends with a_k and y_k denote the maximum length of a increasing subsequence that ends with a_k.
   If there is no decreasing or increasing subsequence of length $n + 1$, then $1 \leq x_k \leq n$ and $1 \leq y_k \leq n$ for all values of k. Consequently, there are at most n^2 distinct ordered pairs (x_k, y_k). However we have a sequence of $n^2 + 1$ ordered pairs associated with each term a_k in the sequence. So, by pigeonhole principle, we have two identical ordered pairs $(x_i, y_i), (x_j, y_j)$, where $i \neq j$. Since terms in the sequence are distinct, we arrive at a contradiction - either $a_i < a_j$ then $x_i > x_j$ or $a_i > a_j$ then $y_i < y_j$.

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4. State well ordering principle. State and Prove Mathematical Induction.
   Solution :
   The well-ordering principle states that every non-empty subset of positive integers contains a least element.

   The principle of Mathematical induction states that:
   Let $S(n)$ denote an open mathematical statement that involves one or more occurrences of the variable n, which represents a positive integer:

   (a) If $S(n)$ is true; and

   (b) If whenever $S(k)$ is true then $S(k + 1)$ is true;

   then $S(n)$ is true for all $n \in Z^+$

   The proof of the above is as stated:
   Let $S(n)$ be such an open statement satisfying (a), (b), and let $F = \{t \in Z^+ | S(t)\ is\ false\}$. We wish to prove that $F = \phi$, so to obtain a contradiction we assume $F \neq \phi$. By well ordering principle F has a least element m. With $m - 1 \notin F$, we have $S(m - 1) = true$. So by condition (b) we have that $S(m - 1) + 1) = S(m) = true$, contradicting that $m \in F$.

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5. What is the condition for a function to be invertible? Explain with an example.

Solution:
For a function f to be invertible, it must be both one-one and onto.

Let $f : A \to B$ be not onto. Then $\exists\ y \in B \ni'\ y$ does not have a pre-image. That is, $\forall x \in A, f(x) \neq y$. Therefore $f^{-1}(y)$ does not exist.
Similarly, if f is not one-one, $\exists\ y \in B \ni'\ f(x_1) = f(x_2) = y$, for some $x_1, x_2 \in A$. Again in this case, $f^{-1}(y)$ does not exist.

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6. Six boxes are coloured red, black, blue, yellow, orange and green. In how man ways can you put 20 identical balls into these boxes such that no box is empty?
Solution:
This can be done in $\binom{(20-6)+6-1}{5}$ ways.
This is the same as the problem of having combinations with repetitions. Since no box is to be empty we have in total $(20-6)$ balls left after placing 1 ball in each box.

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7. Prove that if R is a reflexive relation on set S, then so is any superset of R inside S $\times$ S.
Solution:

Since R is already reflexive, $\forall a \in S, (a, a) \in R$. Hence a superset of R will continue to have the existing elements. Hence R continues to be reflexive.

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8. Let $G = (V, E)$ be a loop free undirected graph. Prove that if G contains no cycle of odd length then G is bipartite.
Solution:

Suppose G has no odd cycles.
Choose any vertex $v \in G$.

Divide G into two sets of vertices like this:
Let A be the set of vertices such that the shortest path from each element of A to v is of odd length;
Let B be the set of vertices such that the shortest path from each element of B to v is of even length.
WLOG, let $v \in B$ and $A \cap B = \phi$.

Suppose $a_1, a_2 \in A$ are adjacent. Then there would be a closed walk of odd length cycle $(v, \ldots, a_1, a_2, \ldots, v)$.

This contradicts our initial supposition that G contains no odd cycles.
So no two vertices in A can be adjacent.
By the same argument, neither can any two vertices in B be adjacent.
Thus A and B satisfy the conditions for $G = (A \cup B, E)$ to be bipartite.

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Section II [10 Marks each]

1. Prove by Induction that $1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$.
Solution:
We see that the series is sum of Harmonic numbers
Let H_j be the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$
Now we prove that :
$H_{2^n} \geq 1 + \frac{n}{2}$
BASIS STEP: $P(0)$ is true, because $H_{2^0} = H_1 = 11 + \frac{0}{2}$
INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is,
$H_{2^k} \geq 1 + \frac{k}{2}$
where k is an arbitrary non-negative integer. We must show that if $P(k)$ is true, then $P(k+1)$, which states that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$, is also true.
So, assuming the inductive hypothesis, it follows that:

$$\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} \cdots \frac{1}{2^{k+1}} \\
&= H_{2^k} + \frac{1}{2^k+1} \cdots \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} \cdots \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k . \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} \\
&= 1 + \frac{k+1}{2}
\end{aligned}$$

Since we see that H_{2^n} is diverging, we can see that $1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$.

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2. Show that

$$1.2.3 + 2.3.4 + 3.4.5 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

.
Solution :

Proof by induction :
$S(1) = 1.2.3 = \frac{1.2.3.4}{4}$, according to the closed form given. Hence, we can conclude $S(1)$ is true.

Induction Hypothesis: Let the given statement be true up to some $k \ni'$ $1.2.3 + 2.3.4 \ldots k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$

Now, we need to prove that $S(k+1)$ is true.
Consider $1.2.3 + 2.3.4 \ldots k(k+1)(k+2) + (k+1)(k+2)(k+3)$

$$
\begin{aligned}
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\
&= (k+1)(k+2)(k+3)\left(\frac{k}{4} + 1\right) \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4}
\end{aligned}
$$

Hence Proved.

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3. You need to choose a password which is at least 6 characters and at most 8 characters in length with an added condition that each character is an uppercase letter or a digit. Also, your password must contain at least one digit. In how many ways can you choose your password?
Solution:
$(36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8)$

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4. Enumerate all possible non-isomorphic graphs on 4 vertices.
Solution :
0 edges: 1 unique graph.
1 edge: 1 unique graph.
2 edges: 2 unique graphs: one where the two edges are incident and the other where they are not incident.
3 edges: 3 unique graphs. One is a 3 cycle with an isolated vertex, and the other two are trees: one has a vertex with degree 3 and the other has 2 vertices with degree 2.
4 edges: 2 unique graphs: a 4 cycle and one containing a 3 cycle.
5 edges: 1 unique graph.
6 edges: 1 unique graph.