Solutions to Test 1

February 5, 2017

1. There are n people present in a room. Prove that among them there are two people who have the same number of acquaintances in the room. (1 mark)

   Answer: Given n people there are two possibilities.

   - Case 1: When there is a possibility of a person having no acquaintances, then the total possible number of acquaintances one can have are $0, 1, 2, 3, ..., n - 2$.
   - Case 2: When each person knows at least one other person, then the possible number of acquaintances one can have are $1, 2, 3, ..., n - 1$.

   It is observable that in both the cases there are $n - 1$ possibilities as the number of acquaintances(pigeon holes) while there are n people(pigeons). So, by pigeonhole principle, there are at least two people who have the same numner of acquaintances.

2. In 2016, there were 35,000 rank holders in some national level entrance exam, with same rank being shared by multiple people. Alice, Bob, Cathy, Dirac and Elisa cleared this exam. In how many ways can these 5 students secure their ranks? (2 marks)

   Answer: As there is a possibility for more than one person to secure a rank, each of Alice, Bob, Cathy, Dirac and Elisa have the possibility of securing one rank out of 35000 possibilities. Hence, for 5 students, the number of ways they can get ranks = $35000 \times 35000 \times 35000 \times 35000 \times 35000$. The product rule applies here, since for each rank secured by a student, there are 35000 possibilites for every other. So, the answer is $(35000)^5$

3. S(8, 5) =?. (2 marks)

   Answer: The value is 1050. 1 mark is alloted for the value and a mark for the formula.

   - Use S(m,n)=S(m-1,n-1) + n S(m-1,n)
   - Use $S(m, n) = (1/n!) \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!(n-j)!} j^n$

4. Give an example of a function $f : A \to B$ and A_1 , $A_2 \subset A$ for which $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. (2 marks)

   Answer: Any function that satisfies the following criteria fetches 2 marks.

5. State and prove the generalized pigeon hole principle.(2 mark)

   Answer: If N objects are place in k boxes, then there must be at least one box with at least $\lceil \frac{N}{k} \rceil$ objects.

   Let us say we have N objects and k boxes and no box have more than $\lceil \frac{N}{k} \rceil - 1$ objects.Then, the total no of objects would be at max

   $k * (\lceil \frac{N}{k} \rceil - 1) < k * (((N/k) + 1) - 1) = k * (N/k) = N$

   So, no of objects would be then less than N. So, a contradiction and hence proved.

6. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games. (2 marks)

   Answer:

   Let, a_i denotes no of games played on the i^{th} day. The team plays the following number of games in that month -

$a_1, a_2, a_3, \ldots a30$. Given that, $\sum_{j=1}^{30} a_i \leq 45$.

Let s_i denote the total number of games played up to the i^{th} day.

$s_i = \sum_{j=1}^{i} a_i$

We observed that at least 1 game is played in each day. So, each $s_i \leq s_i + 1$.

$s_1 < s_2 < s_3 < \ldots s_30 \leq 45$.

Define a new sequence t such that $t_i = s_i + 14$.

$t_1 < t_2 < t_3 < \ldots t_30 \leq 59$.

Since each value in s_i is unique , so each value in t_i is also unique. Howwever, we have a set of 60 numbers.

By pigeon hole principle, there exists at least 2days which have same number of games played.

7. An odd number of people stand in a park at mutually distinct distances. At the same time each person throws a stone at their nearest neighbor, hitting this person. Use mathematical induction to show that there is at least one survivor, that is, at least one person who is not hit by a stone. (4 marks)

   Answer: We see that for 3 people there always exists a pair of who hit each other and the third hits one of them. Thus the third is left without hit. This is because, the pair with the minimum distance will choose to hit each other. Let $p(i)$ denote i people standing in the ground and 1 left without being hit. Let $p(3)$ is the base case. Let us assume that we have the given statement is true for all odd values up to k, where $k = 2x + 1$. Now we need to prove that $p(k + 2)$ is also true.

   - case 1: there exist at least one pair who hit each other. In this case we remove the pairs and by inducting on the remaining people, 1 person is surely left out.
   - case 2: No pair exists who hits each other. Therefore we obtain a odd length cycle. Without loss of generality we can represent this cycle as $1 \to 2 \to 3 \ldots n - 1 \to n \to 1$, where $x \to y$ denotes that x hits y. Let $d(x, y) = d(y, x)$ denote the distance between person x and person y. Given that $n \to 1$, that is n hit 1, we can say that:
     $d(1, n) = d(n, 1) < d(2, n), d(3, n)$.................$d(n - 1, n)$
     Since 1 hit 2 we can say that $d(n, 1) > d(1, 2)$.

     Due to the cycle we can also say that: $d(1, 2) > d(2, 3) > d(3, 4), \ldots d(n - 1, n) > d(n, 1) > d(1, 2)$ Thus, a contradiction and it can not be odd length cycle.There must be one pair who hit each other hence proved.